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QUATERNIONS.

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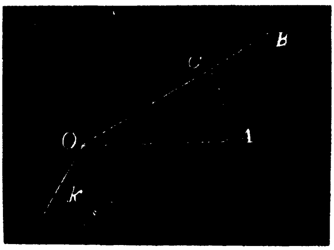
(Continued from page 38.)

MULTIPLICATION AND DIVISION OF RECTANGULAR VECTORS.

14. THE addition and subtraction of vectors, the principles of which were given in the preceding Art., are useful in determining the relation between points. This includes a large and important field in geometry, some of the problems of which may be classed under the general heads of Symmetry, Transversals, Anharmonic Ratio, Mean Point, &c.* We next proceed to compare one vector with another.

15. Conceive two vectors in space, not parallel, and of unequal lengths. Through some point draw two lines parallel and equal in length respectively to the two lines, then will these lines, as vectors, be equal respectively to the former ones. Let $OA = \alpha$ and $OB = \beta$ be these vectors, it is required to compare OB with OA, the comparison involving both their relative directions and relative lengths. Proceeding in a practical way, we first revolve the line OA about the end O, or, more properly, about an axis through O perpendicular to the plane of AB, to coincide in direction with OB.



Suppose that A falls at C, then will their relative length be expressed by

$$\frac{\text{length } OB}{\text{length } OA} = \frac{OB}{OC}.$$

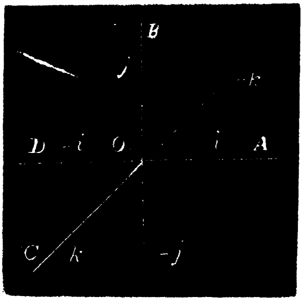
*Hamilton devoted about 30 pages to this part of the subject in his Lectures, and about 90 pages in his Elements of Quaternions—a posthumous work of 762 pages, pub. 1866: Also Tait and Kelland Introduction to Quaternions about 25 pp., and about 30 pp. in the writer's Co-ordinate Geometry.

The problem before us may be stated thus, to find a value for the expression
$$\frac{\text{vector } OB}{\text{vector } OA} = \frac{\beta}{\alpha}.$$

The notation of ordinary algebra not being applicable to this case, a notation and system had to be invented.

16. *Positive rotation* will be considered left-handed,* or in a direction opposite to that of the motion of the hands of a watch, the face of the watch being towards the eye as the observer views the rotation. The UNIT-ANGLE will be taken as a quadrant—or 90°—and *rotation* through a quadrant will be A UNIT-OPERATION in regard to rotation; hence a rotation of a part of a quadrant will be a part of a unit operation, and more than a quadrant, a multiple unit.

17. Beginning with the simplest case, we take three mutually perpendicular lines† of equal length passing through a point, and let their common length be called a unit; then will each line serve as an axis about which either of the others may be revolved. Let $OA = i$, be positive to the right, $OB = j$, positive upwards, and $OC = k$, positive in front of the plane of AOB; then will i, j, k‡ constitute a system of three mutually perpendicular unit-vectors. OA being positive to the right, AO will be negative, and may be represented by OD, drawn to the left of O on AO prolonged; and, making its length unity, it will be represented by $-i$. Similarly for $-j$ and $-k$. Let the line OA be revolved positively



*Hamilton considered right-handed rotation as *positive*, but truly remarked that either direction may be assumed as such. I choose the former because it agrees with the conventional method of generating an angle in Trigonometry, Analytical Geometry and Mechanics.

†It is impossible to have more than three mutually perpendicular lines in a set having a common point, and hence all problems involving four or more rectangular dimensions of space are necessarily hypothetical. But such problems may properly be the subject of analysis when laws are assigned to the hypothetical conditions.

Thus, if the radius be unity, it is impossible to have a cosine equal to 2, but by assuming that the laws of trigonometry apply to hypothetical (or impossible) cases in that science, an *expression* may be found for each of the trigonometrical functions when $\cos x = 2$. For we will have $\cos^2 x + \sin^2 x = 1$; $\therefore \sin x = \sqrt{-3}$, and similarly for other values. This *result* shows that the assumption involves an impossible condition—one contrary to the *limitations* of the science. The principle is the same as that contained in the old arithmetical problem,

"If one-third of six be three, What will a fourth of twenty be?"

Hamilton, at one time, sought to connect *calculation* with *Geometry*, through some undiscovered *extension*, to space of three dimensions. Lectures, preface, p. (31).

‡This is Hamilton's notation, Lectures, p. 59. The letters i, j, k, are, by common consent, appropriated for this purpose. The letter i had long been used by mathematicians to represent the even root of negative unity.

(that is, left-handed) through a unit angle, about $+k$ as an axis, to coincide with $+j$. It is required to devise an expression which will iuvolve simultaneously the unit angle and the three vectors, and which will call to mind the nature of the operation. Conceive that k is the agent producing the rotation and that $+k'$, or simply k, represents the fact that it *produces a unit operation*, and writing "$=$" for *produces*, we have

$$k = \frac{j}{i}. \tag{3}$$

This expression is read 'k equals j divided by i, or i divided into j' but not 'k equals i into j', for this last expression has been reserved for multiplication.* This expression, though called *an equation*, is not algebraic in the ordinary sense of that science.† It is a vector equation and must be multiplied as such. Every notation carries whatever is assigned to it; hence every equation between three vectors of the *form* of (3) expresses the fact that—the left member represents a vector perpendicular to the plane of the other two, about which as an axis the vector represented by the denominator of the right member is turned in a positive direction through a quadrant to coincide with the vector of the numerator.

In this manner each of the following equations may be interpreted.

$$k = \frac{-j}{-i}, \qquad k = \frac{i}{-j}, \qquad k = \frac{-i}{j}, \tag{4}$$

$$i = \frac{k}{j}, \qquad i = \frac{-j}{k}, \text{ \&c.} \qquad j = \frac{i}{k}, \qquad j = \frac{-k}{i}, \text{ \&c.}$$

If the vector axis be $-k$, positive rotation will be from j towards i &c.; hence

$$-k = \frac{-j}{i}, \qquad -k = \frac{i}{j}, \text{ \&c.,} \tag{5}$$

which results are the same as a negative rotation about $+k$.

A comparison of equations (3) and (5) shows that a reciprocal of the fraction *changes the sign* of the vector axis, instead of producing its reciprocal.

*Hamilton sought to preserve not only the symbols of algebra—as letters and signs—but also, in a measure, its language in the use of terms—as addition, subtraction, multiplication, division, raising to powers, and extracting roots. The enlarged and modified use of a symbol, as occasion demands, is a process constantly going on in mathematics, and is classed as "The principle of extension by the removal of restrictions." This was recognized by Hamilton—See Lecturs p. (15), foot note.

This enlargement of the signification of terms and symbols is much more desirable than the introduction of new terms and signs to represent every new shade of meaning; for it not only avoids the necessity of taxing the memory in using them but when the analysis degenerates into the more ordinary operations, the necessity of transforming a new into an ordinry notation is avoided.

†It really involves *four* units, *one* of length and *three* of relative direction.

18. Suppose that k operates upon i turning it in one operation through two unit angles, it will bring i into the position $-i$. Then will the right member of the equation become $i \div -i = -1$, but what shall be the form of the left member? It must be $2k$, ${}^2k$, ${}_2k$, k^2, or some other form. Instead of making an arbitrary choice we observe that multiplying (3) and (4) together, member by member, will give minus 1 (as it should, since it is simply cobmining two successive unit operations) and we have

$$k.k = \frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i} = -1.$$

The left member is of the *form* of the multiplicution of two equal factors in algebra; hence borrowing (or extending) the notation of that science for this case,* we have

$$k^2 = -1, \tag{6}$$

which is read in the usual way 'k square equals minus 1'—or more fully, 'the square of a unit vector produces minus unity'. In other words, it reverses the *direction* of a line.

19. The index (or exponent) expresses the number of units through which a line is revolved, *the number of quadrants being equal to the units in the index.* Hence k^3 implies a rotation through three quadrants about k as an axis; k^4, four quadrants; $k^{1/2}$, one half a quadrant or 45°; $k^{1/3}$, a rotation of 30°; k^{-1}, a rotation in a *negative* direction through a quadrant, $-k^{2/3}$, a *positive* rotation of 60° about a negative vector axis; $-k^{-1\frac{1}{2}}$, a *negative* rotation of 135° about a *negative* vector axis; and generally k^t represents a rotation through t quadrants, or parts of a quadrant, about a vector axis; where t may be positive or negative, entire or fractional, and k a unit vector, positive or negative. If θ be an angle measured by the subtended arc, we have $t = \theta \div \frac{1}{2}\pi$, and

$$k^t = k^{\frac{2\theta}{\pi}}. \tag{7}$$

20. A VERSOR† is a representative of rotation. The axis about which rotation takes place is taken as a representative of the agent producing the rotation, and the angle as the amount of rotation produced by the agent; and the axis and rotation combined is called *the versor*, and may be represented as in equation (7) above. If the rotation be through a quadrant, it is called a *quadrantal versor*, hence i, j, k, or any other unit vector with unity for an exponent, is a quadrantal versor. The letter U‡ placed before

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*One of the peculiarities of Hamilton's style may be noticed by carefully reading Articles 75, 81, and 85, of his Lectures. It will be observed that he uses $\sqrt{-1}$ before he introduces the exp. 2; and when he introduces the latter (p. 81) he simply says "or using the EXP. 2"&c.

†From a Latin word signifying that which turns about. ‡Lectures, pp. 88, 118.

an expression implies that the rotation, or the act of version, is to be considered. Thus, $U(\beta \div \alpha)$ implies the turning of vertor α to coincide in direction with vector β, hence if θ be the angle between α and β we have

$$U\frac{\beta}{\alpha} = k^{2\theta \div \pi *}, \tag{8}$$

where θ may be positive or negative and k also positive or negative. The letter U may also be used to denote *a unit vector*; thus $U\beta$ is a unit vector parallel to vector β. As a versor it implies that β has been turned from some arbitrary direction into the given one. Hence we may write

$$\beta = T\beta . U\beta . \dagger$$

The versor of a number is merely a sign, + or —, but as multiplying by unity does not change the product, it may be written +1 or —1; thus

$$U.3b = +1, \qquad U(-\sqrt{2}) = -1.$$

From equation (6) we have

$$k = -\frac{1}{k} = -(k^{-1}); \ddagger \tag{9}$$

that is, a vector equals minus the reciprocal of the same vector.

21. From equation (6), by taking one-half the exponents of both sides and using the radical sign as in algebra, we have

$$k = (-1)^{\frac{1}{2}} = \sqrt{-1}. \tag{10}$$

This is the *form* of the simplest imaginary of algebra, but it is not to be interpreted algebraically. The right member is simply a particular expression for the left member,** and hence implies that a line is rotated through a quadrant. The same expression may not only be found for i and j,†† but the line to be rotated will also disappear from the expression. Hence, Hamilton called the expression $\sqrt{-1}$, when separated from i, j, k, or any other axis, "an INDETERMINATE VECTOR-UNIT, or a unit vector with indeterminate direction."‡‡ In this system it is perfectly real,‡′ but in algebra it is impossible and is properly called imaginary‡″—it does not represent any real line.‡‴ On account of its indeterminate character it is rarely used in this science.

The expression $k^2 = -1$ has an infinite number of real roots, since it is applicable to an infinite number of pairs of rectangular vectors in a plane perpendicular to vector k. The corresponding case in algebra is furnished by the equation of the sphere

$$r^2 = x^2 + y^2 + z^2,$$

*Lectures, p. 88. †Ibid., p. 114. ‡Ibid., pp. 122, 123, 126. **It simply denotes a *certain* vector unit, Ibid., p. 397. ††"Every unit vector is regarded as one of the square roots of negative unity." Preface, p. (61), foot note. ‡‡Ibid., p. 178. ‡′Ibid., p. 180. ‡″Ibid., pp. 635–36. ‡‴Ibid., p. 638.

which, being true and real for all values of x, y and z less than r, gives an infinite number of values of r, all equal in length, but different in position. The equation $k^2 = -1$, when k has a fixed position, is the equation of a circumference whose radius is unity; but as the position of k is indeterminate, it is the equation of a sphere whose radius is unity.*

Although we have

$$i^2 = j^2 = k^2 = -1,$$

it does not follow that $\quad i = j = k,$

and it is evident from the figure and the definition of a vector that they cannot be equal, for they necessarily differ in *direction*. The case is similar to one in algebra, where we have

$$x^0 = y^0 = z^0 = 4^0 = 1,$$

from which it does not follow that

$$x = y = z = 4 = 1;$$

neither does it follow from the equation

$$0 . x = 0 . y \text{ that } x = y.$$

These are cases where the general values of the quantities differ, but which reduce to an identity under particular forms of expression.

22. The preceding operations are called The Division of Vectors; we now proceed to consider their multiplication. If k operate upon i producing j, as before described, and then the lines be multiplied together, we may express the operation thus

$$k i = j, \tag{11}$$

which is read 'k into i equals j' or 'i multiplied by k equals j'; but not 'i into k' nor 'k multiplied by i'.† This compared with equation (3) shows that the expression for division may be changed to multiplication by clearing the former of fractions as in algebra, being particular however to place the versor as the *first* factor, for reasons which will soon appear. We thus see that the multiplication and division of rectangular unit vectors are intrinsically the same.‡ An examination of the figure shows that, in the last equation, j may be considered as the versor turning k into i.

If k turns i through two unit angles, the expression becomes

$$k^2 i = -i; \therefore k^2 = -1, \text{ and } k = \sqrt{-1},$$

as before. If the length of the vector be a we have

$$(ak)(ak) = a^2k^2 = -a^2,$$

that is, the square of a vector equals minus the square of a line of equal length.‡′ This furnishes one mode of passing from a vector to a line.

*Lectures, p. 181. †Ibid., p. 37. Hamilton interprets this eq'n thus;—a quadrantal rotation about i from k to $-j$ followed by a quadrantal rotation about $-k$ from $-j$ to i is the same as a single quadrantal rotation about j from k to i. ‡Ibid., pp. 90, 91. ‡′Ibid., p. 81.

23. If i operate upon k positively, we have (see the figure)

$$ik = -j, \tag{12}$$

which compared with (11) shows that ki does not equal ik, but that we have

$$ki = -ik.$$

This is called the *non-commutative* principle of the multiplication of vectors—by which is meant that the product is not the same when the factors are interchanged.* This property draws a sharp line of distinction between this science and ordinary algebra.†

The signs $+$ and $-$ are commutative; for an exam. of the fig. shows that

$$(+i)(-j) = -k = (-i)(+j) = -ij.$$

If both sides of the equation $ij = k$, be multiplied into k, we have

$$ijk = k^2 = -1, \text{ also } i^2 = j^2.‡$$

If the cyclical order be preserved the result will remain the same, but if that order be deranged the sign of the result will be changed, and we will have

$$ijk = jki = kij = -kij = -ijk = \quad jki = -1.‡‡$$

24. *The associative principle* consists in the grouping of the factors in different ways. It may be shown that this principle is the same in this system as in algebra,** so that we have

$$ai.jk = a.ij.k = aijk.††$$

25. *The distributive principle* consists in making the product of a factor into the sum of two or more quantities equal to the sum of the products of the factors into each of the quantities separately. This principle is also the same in this system as in algebra; hence we have

$$a(2\beta + b\gamma) = 2a\beta + ba\gamma.$$

(To be continued.)

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*Hamilton labored to retain the *commutative* principle but was compelled to abandon it in order to establish his system. Lectures, preface, pp. (43)-(46). ‡‡Ibid., pp. 208, 237, 333.

†There are however many instances in arithmetical and algebraic analysis in which a change of place of the characters or symbols produces a change in the result, as will be seen by comparing the following pairs of expressions:—$\log\cos x$, $\cos\log x$; $\sin\tan^{-1}x$, $\tan^{-1}\sin x$; $\log\sec^{-1}x$, $\sec^{-1}\log x$; a^x, x^a; $\frac{3}{7}$, $\frac{7}{3}$; 45, 54.

‡The establisoment of this equation was considered as the foundation of the syst. p. (46)

**The associative principle is not a necessity. Thus, in the octinomials of Caley and Graves, of the form $a+ib+jc+kd+le+mf+ng+oh$, where i, j, &c., are each $\sqrt{-1}$, we have

$$i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = ijk = ilm = ino = jln = jmo = klo = knm.$$

But $i.jl = in = -0 = -kl = -ij.l$. Phil. Mag., May, 1845, p. 210. Trans. R. Irish A. Vol. XXI, part II, pp. 338, 339. ††Lectures, pp. 199, 289, 300.